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TECHNICAL NOTES

Errors in one-dimensional fin optimization problem for convective heat transfer

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INTRODUCTION

Finned surfaces have been in use for a long period as a heat dissipation mechanism. The optimization of fins is of major importance in the thermal design of a heat exchanger. The design criteria of fins are different for various applications, but the primary concern is performance, weight and cost. In the optimization of a fin with uniform cross section, it is usual to obtain the optimum dimensions of a fin which dissipates the maximum heat for a given mass.

Recently, numerous works [1–7] have been conducted to optimize the dimensions of fins in convective as well as boiling heat transfer. All these studies were based on a one-dimensional heat conduction assumption and very few analyses included a convective boundary condition at the fin tip. The one-dimensional approach is convenient, but may be in error for certain physical conditions. Irey [8] and Lau and Tan [9] showed that the one-dimensional assumption for a fin is valid only for a transverse Biot number much less than unity. Aparecido and Cotta [10] proposed a modified one-dimensional formulation to improve the accuracy of the one-dimensional solution. Whether the criterion established by Irey [8] and Lau and Tan [9] is valid for the optimum dimensions of fins or not still remains to be determined.

The optimum dimensions of a two-dimensional longitudinal rectangular fin were investigated analytically by Look and Kang [11]. In their work, considering a periodic root temperature and different constant convection coefficients at top, bottom and tip surfaces of a fin, the relationship between the aspect ratio for 0.99 Q_{opt} and fin's usefulness was developed. Razelos and Georgiou [12] suggested that the measure of the heat transfer augmentation can be expressed by effectiveness or the removal number which can serve as a criterion in assessing the solutions of the classical one-dimensional method. Lately, Chung and Iyer [13] have solved the same fin problem with uniform base temperature and heat transfer from tip. Their results are very complicated for use due to the consideration of so many parameters.

The purpose of this study is first to determine the optimum dimensions of a one-dimensional longitudinal rectangular fin and a cylindrical pin fin. Considering heat transfer from the fin tip, the corresponding two-dimensional problems are solved to check the validity of the one-dimensional models. To circumvent the tedious procedures in solving the two-dimensional fin problems, a modified one-dimensional solution for fin optimization is also presented. In addition, special cases of fins with insulated tips are also taken into account.

ANALYSIS

In this study, assumptions are made of steady-state heat conduction through the fin, constant thermal conductivity of the fin material, no heat source in the fin, and uniform ambient temperature. In addition, the temperature of the base of the fin is assumed to be uniform. The circumferential surface heat-transfer coefficient, h, and the heat-transfer coefficient at the free end of the fin, h_e, are taken as constant.

Rectangular fin

In this optimization work of a longitudinal rectangular fin, the principle is to find a fin with maximum heat dissipation for a given fixed profile area, heat transfer coefficient, and thermal conductivity. The two-dimensional solution of this fin problem may be readily found in [10, 14]. Differentiating the heat transfer rate Q with respect to $\alpha(=l/b)$ and setting the result equal to zero yields

$$\sum_{n=1}^{\infty}\frac{\lambda_n^2\sin^2(2\lambda_n)}{[2\lambda_n+\sin(2\lambda_n)]^3}\cdot\left[\frac{\varepsilon+\cot(\lambda_n)\tanh(\phi_o)}{\cot(\lambda_n)+\varepsilon\tanh(\phi_o)}\right]$$
$$+\sum_{n=1}^{\infty}\frac{\lambda_n\sin^2(\lambda_n)}{[2\lambda_n+\sin(2\lambda_n)]^2}\cdot\frac{\operatorname{sech}^2(\phi_o)}{[\cot(\lambda_n)+\varepsilon\tanh(\phi_o)]^2}$$
$$\cdot\left\{\varepsilon\cot(\lambda_n)-\frac{\phi_o}{2\lambda_n}[4\lambda_n+\sin(2\lambda_n)]\right.$$
$$\left.\cdot[\cot^2(\lambda_n)-\varepsilon^2]\right\}=0 \qquad (1)$$

where $\varepsilon = h_e/h$ and $\phi_o = 2\lambda_n\cdot\alpha_{\text{opt}}$. Note that the eigenvalues λ_n can be obtained from

$$2\lambda_n\cdot\tan\lambda_n - B_a\cdot\alpha_{\text{opt}}^{-1/2} = 0. \qquad (2)$$

For any given B_a, the optimum aspect ratio of the fin can be determined from equations (1) and (2). An iterative technique is used to find α_{opt}. Initially, an α_{opt} is guessed at a specified B_a, where all the λ_n's can be obtained from equation (2). For a given ε, the optimum aspect ratio and the calculated eigenvalues are then substituted into equation (1). The procedure is continued until equation (1) is satisfied within a tolerance of 10^{-7}.

In the corresponding one-dimensional solutions, the maximum heat dissipation is obtained by taking the deriva-

NOMENCLATURE

A	profile area of a rectangular fin [m^2]
B_a	Biot number, $hA^{1/2}/k$
B_i	Biot number, $hb/(2k)$
B_v	Biot number, $hV^{1/3}/k$
b	fin thickness (or diameter) [m]
h, h_e	heat transfer coefficients at lateral and tip surface [W m^{-2} K^{-1}]
k	thermal conductivity of fin material [W m^{-1} K^{-1}]
l	fin length [m]
Q	dimensionless heat transfer rate, $q/(k\theta_b W)$ for a rectangular fin and $qh/(4\pi k^2\,\theta_b)$ for a cylindrical fin
q	heat transfer rate through fin base [W]
V	volume of a cylindrical fin [m^3]
W	width of a longitudinal rectangular fin [m].

Greek symbols

α	aspect ratio, l/b
ε	ratio of heat transfer coefficients of fin tip to lateral fin surface, h_e/h
θ	fin temperature in excess of ambient [°C]
λ_n	eigenvalues.

Subscripts and superscripts

b	fin base
dim	dimension
e	fin tip
o	optimum
opt	optimum
*	percentage error.

tive of fin's heat transfer with respect to the aspect ratio and setting it equal to zero. It gives

$$2\omega_o^{1/2}+(1+\omega_o)\tanh(\psi_o)+3\psi_o(\omega_o-1)\,\mathrm{sech}^2(\psi_o)=0 \tag{3}$$

where $\omega_o=\varepsilon^2/2\cdot B_a\cdot\alpha_{opt}^{-1/2}$ and $\psi_o=\sqrt{(2B_a)}\cdot\alpha_{opt}^{3/4}$.

To extend the applicability range of the one-dimensional formulation, a modified one-dimensional solution [10] is introduced. The optimum relationship is derived as

$$\begin{aligned}&8\varepsilon B_a\sqrt{(\alpha_{opt})}+[4\mu_o+\varepsilon^2B_a^{3/2}(8\sqrt{(\alpha_{opt})}+B_a)^{1/2}]\tanh(\mu_o)\\&\quad+[4\mu_o^2-32B_a\alpha_{opt}^{3/2}+\varepsilon B_a^2+2\varepsilon^2B_a^2(6\alpha_{opt}\\&\quad+B_a\sqrt{(\alpha_{opt})})]\,\mathrm{sech}^2(\mu_o)=0\end{aligned} \tag{4}$$

where $\mu_o=4\alpha_{opt}\sqrt{B_a}(8\sqrt{(\alpha_{opt})}+B_a)^{-1/2}$.

Apparently, the optimum aspect ratio of the one-dimensional rectangular fin can be obtained from equations (3) and (4) for any given B_a and ε.

Cylindrical pin fin

The heat transfer rate of a cylindrical fin is optimized for a fixed fin volume and given constant thermal properties. Following the same mathematical procedures as those given in the case of a rectangular fin yields

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\lambda_n^3}{(1+\Gamma_n^2)^2\Gamma_n}\cdot\frac{\mathrm{sech}^2(\phi_o)}{[\Gamma_n+\varepsilon\tanh(\phi_o)]}\cdot\left[v_o\Gamma_n\left(\frac{\Gamma_n}{\lambda_n}-3\Gamma_n^2-3\right)\right.\\&\quad\left.+\varepsilon\left(1-\frac{1}{\lambda_n\Gamma_n}+\frac{1}{\Gamma_n^2}\right)+\frac{B_v^3\varepsilon^2}{\pi\lambda_n^2}\left(3-\frac{\Gamma_n}{\lambda_n}+3\Gamma_n^2\right)\right]\\&\quad+\sum_{n=1}^{\infty}\frac{\lambda_n^3}{(1+\Gamma_n^2)^3\Gamma_n^4}\left[\frac{\varepsilon+\Gamma_n\tanh(\phi_o)}{\Gamma_n+\varepsilon\tanh(\phi_o)}\right]\\&\quad\cdot\left(3\Gamma_n^4-\frac{3\Gamma_n^3}{\lambda_n}+4\Gamma_n^2-\frac{\Gamma_n}{\lambda_n}+1\right)=0\end{aligned} \tag{5}$$

where $\Gamma_n=J_o(\lambda_n)/J_1(\lambda_n)$ and $\phi_o=2\lambda_n\cdot\alpha_{opt}$. The eigenvalues are evaluated from $(2\pi\alpha_{opt})^{1/3}\cdot\lambda_n\cdot J_1(\lambda_n)-B_v\cdot J_o(\lambda_n)=0$. The solution procedures are identical to those of a rectangular fin case.

For one-dimensional solutions, by letting $\mathrm{d}Q/\mathrm{d}\alpha=0$, one obtains

$$\begin{aligned}&3\varepsilon\sqrt{B_v}\left(\frac{4}{\pi\alpha}\right)^{1/6}+3\left[1+\frac{\varepsilon^2B_v}{4}\left(\frac{4}{\pi\alpha}\right)^{1/3}\right]\tanh(\xi_o)\\&\quad+\left[5\varepsilon^2\left(\frac{\alpha B_v^3}{\pi}\right)^{1/2}-\varepsilon\sqrt{B_v}\left(\frac{4}{\pi\alpha}\right)^{1/6}-5\xi_o\right]\mathrm{sech}^2(\xi_o)=0\end{aligned} \tag{6}$$

for classical one-dimensional formulation and

$$\begin{aligned}&4\varepsilon\sqrt{3}B_v^{1/2}(9v_o+B_v)+[3(4v_o+\varepsilon^2B_v)\sqrt{(6v_o+B_v)}\\&\quad+36v_o^2/\sqrt{(6v_o+B_v)}+\varepsilon^2B_v(6v_o+B_v)^{3/2}/v_o]\tanh(\zeta_o)\\&\quad+12\sqrt{3}B_v^{1/2}[12\alpha v_o^2/(6v_o+B_v)+\varepsilon^2B_v\alpha_{opt}(5v_o+B_v)/v_o\\&\quad-12\alpha_{opt}v_o-\varepsilon v_o]\,\mathrm{sech}^2(\zeta_o)=0\end{aligned} \tag{7}$$

for a modified one. Note that in equations (6) and (7) ξ_o and ζ_o are defined as $\xi_o=2\sqrt{B_v}(4\alpha_{opt}^5/\pi)^{1/6}$ and $\zeta_o=4\alpha_{opt}\sqrt{(3B_v/(6v_o+B_v))}$. It is apparent that, once B_v and ε are given, the optimum aspect ratio, α_{opt}, can be evaluated from equations (6) and (7).

RESULTS AND DISCUSSION

It is worthwhile to note that there are design restrictions for optimum fins with heat transfer dissipated from tips for both of rectangular fins and cylindrical pin fins. A maximum B_a or B_v, denoted as $(B_a)_{max}$ or $(B_v)_{max}$, is found for $\varepsilon>0$. No optimum aspect ratios of fins are found for $B_a>(B_a)_{max}$ or $B_v>(B_v)_{max}$. For design consideration, the dependence of $(B_a)_{max}$ and $(B_v)_{max}$ on ε is given in Fig. 1. Note that $(B_a)_{max}$ and $(B_v)_{max}$ are pretty small at a larger ε and increase with decreasing ε for the proposed three solutions. The value of $(B_a)_{max}$ or $(B_v)_{max}$ tends to infinity as ε approaches zero. It is also shown that the maximum values of B_a and B_v evaluated from both of the one-dimensional formulations are a little higher than that from the two-dimensional one. The discrepancies are insignificant at a larger ε but are pronounced especially at a smaller ε. No optimum dimensions ratios of fins are found in the regions above the six given curves. Thus, it is obvious that the design from two-dimensional case imposes more severe restrictions than from the one-dimensional one.

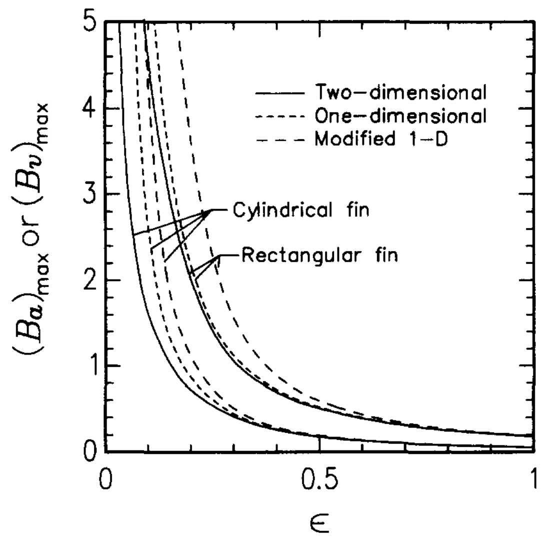

Fig. 1. Dependence of $(B_a)_{max}$ or $(B_v)_{max}$ on ε.

To show the differences between one-dimensional results and two-dimensional ones, it is convenient to define

$$\delta^* = \frac{(\delta_{opt})_{two\text{-}dim} - (\delta_{opt})_{one\text{-}dim}}{(\delta_{opt})_{two\text{-}dim}} \% \tag{8}$$

where δ may be α or Q. The comparison of α_{opt} and the corresponding Q_{opt} from one and two-dimensional analyses is depicted in the upper and lower parts of Fig. 2. Note that the end points of curves are $(B_a)_{max}$. For very low values of B_a, which also implies very small values of Biot number, the differences among the three solutions are relatively small. It is shown that α^* and Q^* of the one-dimensional and modified one-dimensional analyses are small except that B_a is close to $(B_a)_{max}$. With an increase of B_a, the magnitude of errors of both solutions becomes larger. It should be pointed out that the optimum aspect ratios and corresponding heat transfer rate predicted from modified one-dimensional analyses are very close to the two-dimensional solutions especially for an insulated-tip fin. In the lower part of Fig. 2, the error rate percentages of Q^* for $\varepsilon = 0.7$ and 1 do not clearly show up because the data almost completely coincide with Q^* for $\varepsilon = 0$ and 0.3 and are accurate up to 0.2% for the modified one-dimensional results.

Figure 3 shows the dependence of α_{opt} and α^* on ε for $B_a = 0.05$ and 0.1. In this figure, it is noted that the optimum aspect ratios predicted from modified one-dimensional model almost overlaps the two-dimensional results. In addition, the optimum aspect ratios of rectangular fins decrease with increasing ε. Thus, the usual study of fin optimization problem employing the assumption of an insulated free end overestimates α_{opt}. The overestimated values of α_{opt} become large especially at larger ε. The error rate, α^*, increases with ε for the one-dimensional solutions whereas a very slight effect of ε on α^* is observed for the modified one-dimensional ones. Also note that α^* increases with B_a for a fixed ε for both of the one-dimensional solutions.

For a cylindrical pin fin, the percentage errors of α_{opt} and Q_{opt} vs B_v are presented in the upper and lower parts of Fig. 4. Also, the end points of curves are $(B_v)_{max}$. This figure shows that α^* predicted from the modified one-dimensional method is smaller than that of one-dimensional approach at most of the values of B_v for $\varepsilon = 0$. As for the case of $\varepsilon > 0$, the error rates of both models are mostly small except for the values of B_v close to $(B_v)_{max}$. As a whole, the solutions calculated from the modified one-dimensional mode are more accurate than those from the one-dimensional approach.

For a fin with insulated free end, it was confirmed [8, 9] that the major parameter governing the accuracy of the approximate solutions is the transversal Biot number, B_i. The dependence of α^* and Q^* on B_i is displayed in Fig. 5. It is observed that the error rates of both of the one-dimensional solutions are small at a smaller B_i and increase with B_i. Thus, B_i may also serve as a criterion in evaluating the accuracy of the one-dimensional solutions. In addition, for $B_i \leqslant 1$, the error rates of α_{opt} and Q_{opt} are around 2% for modified one-dimensional solution and are 22% for one-dimensional formulation. This points to the fact that the modified approach reduces the error by an order of magnitude compared to that given by the classical one-dimensional approach. In this case, it is evident that the modified one-dimensional model prevails over the one-dimensional model in the predictions of the aspect ratio and heat transfer rate of fins. Hence, the modified one-dimensional method is also very useful in extending the limits of applicability for the one-dimensional approximation for $\varepsilon = 0$.

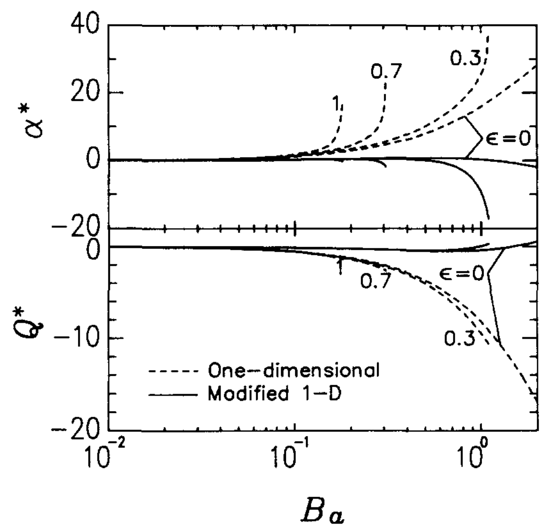

Fig. 2. Percentage error of α_{opt} and Q_{opt} for the one-dimensional and modified one-dimensional solutions of a rectangular fin.

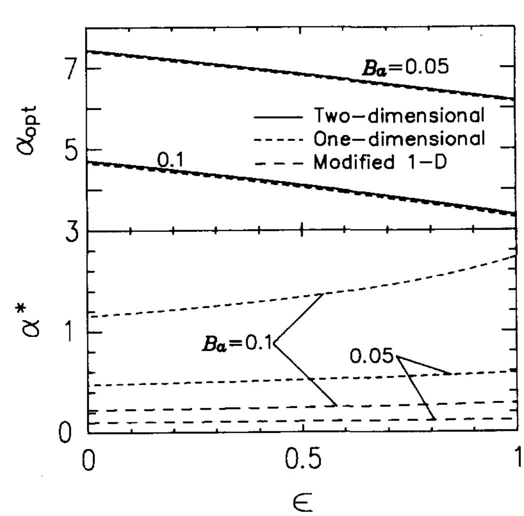

Fig. 3. Dependence of α_{opt} and α^* on ε for $B_a = 0.05$ and 0.1 (rectangular fin).

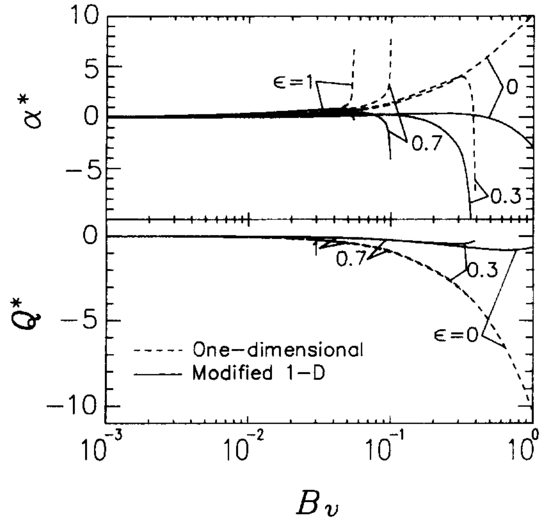

Fig. 4. Percentage error of α_{opt} and Q_{opt} for the one-dimensional and modified one-dimensional solutions of a cylindrical pin fin.

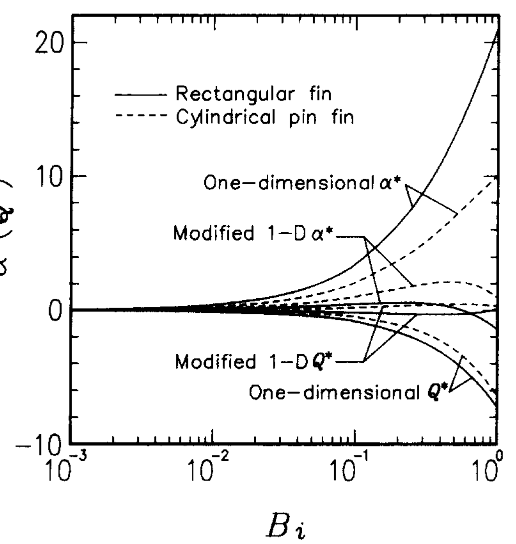

Fig. 5. Percentage error of α_{opt} and Q_{opt} for the one-dimensional and modified one-dimensional solutions of a rectangular fin and a cylindrical pin fin.

CONCLUSIONS

(1) For a fin with convection at tip, there exists a maximum value of B_a. No optimum aspect ratios of fins can be found for $B_a > (B_a)_{max}$. In addition, no design restriction in B_a for an insulated-tip fin. This is also true for a cylindrical pin fin case.

(2) The calculated optimum aspect ratios of both a rectangular fin and a cylindrical fin with heat transfer from the free end are smaller than those of fin with insulated tips. Also, α_{opt} decreases with ε.

(3) In the evaluation of α_{opt} and Q_{opt}, B_a or B_v may serve as a criterion in assessing the validity of one-dimensional approximation. It is shown that the percent error rates of α_{opt} and Q_{opt} grow with B_a or B_v and reach a maximum at $(B_a)_{max}$ or $(B_v)_{max}$ for $\varepsilon > 0$.

(4) In addition to B_a and B_v, B_i is also a good measure in evaluating the accuracies of α_{opt} and Q_{opt} for both rectangular fins and cylindrical pin fins with insulated tips.

(5) In predicting α_{opt} and Q_{opt}, the optimum data derived from modified one-dimensional model may extend B_a, B_v and B_i to a higher value with reasonable accuracy.

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